GEOMETRY OF SURFACES

b3 course 2004

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1 Introduction

This is a course on surfaces. Your mental image of a surface should be something like this:



However we are also going to try and consider surfaces intrinsically, or abstractly, and not necessarily embedded in three-dimensional Euclidean space like the two above. In fact lots of them simply can't be embedded, the most notable being the projective plane. This is just the set of lines through a point in \mathbb{R}^3 and is as firmly connected with familiar Euclidean geometry as anything. It *is* a surface but it doesn't sit in Euclidean space.

If you insist on looking at it, then it maps to Euclidean space like this



- called $Boy's\ surface. This is not one-to-one but it does intersect itself reasonably cleanly.$

A better way to think of this space is to note that each line through 0 intersects the unit sphere in two opposite points. So we cut the sphere in half and then just have to identify opposite points on the *equator*:



... and this gives you the projective plane.

Many other surfaces appear naturally by taking something familiar and performing identifications. A doubly periodic function like $f(x, y) = \sin 2\pi x \cos 2\pi y$ can be thought of as a function on a surface. Since its value at (x, y) is the same as at (x+m, y+n) it is determined by its value on the unit square but since f(x, 0) = f(x, 1)and f(0, y) = f(1, y) it is really a continuous function on the space got by identifying opposite sides:



and this is a torus:



We shall first consider surfaces as topological spaces. The remarkable thing here is that they are completely classified up to homeomorphism. Each surface belongs to two classes – the orientable ones and the non-orientable ones – and within each class there is a non-zero integer which determines the surface. The orientable ones are the ones you see sitting in Euclidean space and the integer is the number of holes. The non-orientable ones are the "one-sided surfaces" – those that contain a Möbius strip – and projective space is just such a surface. If we take the hemisphere above and flatten it to a disc, then projective space is obtained by identifying opposite points on the boundary:



Now cut out a strip:

and the identification on the strip gives the Möbius band:



As for the integer invariant, it is given by the *Euler characteristic* – if we subdivide a surface A into V vertices, E edges and F faces then the Euler characteristic $\chi(A)$ is defined by

$$\chi(A) = V - E + F.$$

For a surface in Euclidean space with g holes, $\chi(A) = 2 - 2g$. The invariant χ has the wonderful property, like counting the points in a set, that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

and this means that we can calculate it by cutting up the surface into pieces, and without having to imagine the holes.

One place where the study of surfaces appears is in complex analysis. We know that $\log z$ is not a single valued function – as we continue around the origin it comes back to its original value with $2\pi i$ added on. We can think of $\log z$ as a single valued function on a surface which covers the non-zero complex numbers:



The Euclidean picture above is in this case a reasonable one, using the third coordinate to give the imaginary part of log z: the surface consists of the points $(re^{i\theta}, \theta) \in \mathbf{C} \times \mathbf{R} = \mathbf{R}^3$ and log $z = \log r + i\theta$ is single-valued. But if you do the same to $\sqrt{z(z-1)}$ you get



a surface with self-intersections, a picture which is not very helpful. The way out is to leave \mathbf{R}^3 behind and construct an abstract surface on which $\sqrt{z(z-1)}$ is singlevalued. This is an example of a *Riemann surface*. Riemann surfaces are always orientable, and for $\sqrt{z(z-1)}$ we get a sphere. For $\sqrt{z(z-1)(z-a)}$ it is a torus, which amongst other things is the reason that you can't evaluate

$$\int \frac{dx}{x(x-1)(x-a)}$$

using elementary functions. In general, given a multi-valued meromorphic function, the Euler characteristic of the Riemann surface on which it is defined can be found by a formula called the Riemann-Hurwitz formula.

We can look at a smooth surface in Euclidean space in many ways – as a topological space as above, or also as a *Riemannian manifold*. By this we mean that, using the Euclidean metric on \mathbb{R}^3 , we can measure the lengths of curves on the surface.



If our surface is not sitting in Euclidean space we can consider the same idea, which is called a Riemannian metric. For example, if we think of the torus by identifying the sides of a square, then the ordinary length of a curve in the plane can be used to measure the length of a curve on the torus:



A Riemannian metric enables you to do much more than measure lengths of curves: in particular you can define areas, curvature and geodesics. The most important notion of curvature for us is the Gaussian curvature which measures the deviation of formulas for triangles from the Euclidean ones. It allows us to relate the differential geometry of the surface to its topology: we can find the Euler characteristic by integrating the Gauss curvature over the surface. This is called the *Gauss-Bonnet theorem*. There are other analytical ways of getting the Euler characteristic – one is to count the critical points of a differentiable function.

Surfaces with constant Gaussian curvature have a special role to play. If this curvature is zero then locally we are looking at the Euclidean plane, if positive it is the round sphere, but the negative case is the important area of hyperbolic geometry. This has a long history, but we shall consider the concrete model of the upper half-plane as a surface with a Riemannian metric, and show how its geodesics and isometries provide the axiomatic properties of non-Euclidean geometry and also link up with complex analysis. The hyperbolic plane is a surface as concrete as one can imagine, but is an abstract one in the sense that it is not in \mathbb{R}^3 .

2 The topology of surfaces

2.1 The definition of a surface

We are first going to consider surfaces as topological spaces, so let's recall some basic properties:

Definition 1 A topological space is a set X together with a collection \mathcal{T} of subsets of X (called the 'open subsets' of X) such that

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
- if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;
- if $U_i \in \mathcal{T}$ $\forall i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$.
- X is called Hausdorff if whenever $x, y \in X$ and $x \neq y$ there are open subsets U, V of X such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.
- A map $f: X \to Y$ between topological spaces X and Y is called continuous if $f^{-1}(V)$ is an open subset of X whenever V is an open subset of Y.
- $f: X \to Y$ is called a homeomorphism if it is a bijection and both $f: X \to Y$ and its inverse $f^{-1}: Y \to X$ are continuous. Then we say that X is homeomorphic to Y.
- X is called compact if every open cover of X has a finite subcover.

Subsets of \mathbf{R}^n are Hausdorff topological spaces where the open sets are just the intersections with open sets in \mathbf{R}^n . A surface has the property that near any point it looks like Euclidean space – just like the surface of the spherical Earth. More precisely:

Definition 2 A topological surface (sometimes just called a surface) is a Hausdorff topological space X such that each point x of X is contained in an open subset U which is homeomorphic to an open subset V of \mathbb{R}^2 .

X is called a closed surface if it is compact.

A surface is also sometimes called a 2-manifold or a manifold of dimension 2. For any natural number n a topological n-manifold is a Hausdorff topological space X which is locally homeomorphic to \mathbb{R}^n .

Remark: (i) The Heine-Borel theorem tells us that a subset of \mathbf{R}^n is compact if and only if it is *closed* (contains all its limit points) and *bounded*. Thus the use of the terminology 'closed surface' for a compact surface is a little perverse: there are plenty of surfaces which are closed subsets of \mathbf{R}^3 , for example, but which are not 'closed surfaces'.

(ii) Remember that the image of a compact space under a continuous map is always compact, and that a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

Example: The sphere. The most popular way to see that this is a surface according to the definition is stereographic projection:



Here one open set U is the complement of the South Pole and projection identifies it with \mathbf{R}^2 , the tangent plane at the North Pole. With another open set the complement of the North Pole we see that all points are in a neighbourhood homeomorphic to \mathbf{R}^2 .

We constructed other surfaces by identification at the boundary of a planar figure. Any subset of the plane has a topology but we need to define one on the space obtained by identifying points. The key to this is to regard identification as an *equivalence relation*. For example, in constructing the torus from the square we define $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ and every other equivalence is an equality. The torus is the set of *equivalence classes* and we give this a topology as follows:

Definition 3 Let \sim be an equivalence relation on a topological space X. If $x \in X$ let $[x]_{\sim} = \{y \in X : y \sim x\}$ be the equivalence class of x and let

$$X/\sim = \{ [x]_{\sim} : x \in X \}$$

be the set of equivalence classes. Let $\pi : X \to X/\sim$ be the 'quotient' map which sends an element of X to its equivalence class. Then the quotient topology on X/\sim is given by

$$\{V \subseteq X/\sim: \pi^{-1}(V) \text{ is an open subset of } X\}$$

In other words a subset V of X/\sim is an open subset of X/\sim (for the quotient topology) if and only if its inverse image

$$\pi^{-1}(V) = \{x \in X : [x]_{\sim} \in V\}$$

is an open subset of X.

So why does the equivalence relation on the square give a surface? If a point lies inside the square we can take an open disc around it still in the interior of the square. There is no identification here so this neighbourhood is homeomorphic to an open disc in \mathbf{R}^2 . If the chosen point lies on the boundary, then it is contained in two half-discs D_L, D_R on the left and right:



We need to prove that the quotient topology on these two half-discs is homeomorphic to a full disc. First take the closed half-discs and set $B = D_L \cup D_R$. The map $x \mapsto x + 1$ on D_L and $x \mapsto x$ on D_R is a continuous map from B (with its topology from \mathbf{R}^2) to a single disc D. Moreover equivalent points go to the same point so it is a composition

$$B \to X/\sim \to D.$$

The definition of the quotient topology tells us that $B/\sim \to D$ is continuous. It is also bijective and B/\sim , the continuous image of the compact space B, is compact so this is a homeomorphism. Restrict now to the interior and this gives a homeomorphism from a neighbourhood of a point on the boundary of the square to an open disc.

If the point is a corner, we do a similar argument with quadrants.

Thus the torus defined by identification is a surface. Moreover it is closed, since it is the quotient of the unit square which is compact.

Here are more examples by identification of a square:

• The sphere



• Projective space



• The Klein bottle



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• The Möbius band



The Möbius band is not closed, as the dotted lines suggest. Here is its rigorous definition:

Definition 4 A Möbius band (or Möbius strip) is a surface which is homeomorphic to

$$(0,1) \times [0,1] / \sim$$

with the quotient topology, where \sim is the equivalence relation given by

$$(x,y) \sim (s,t)$$
 iff $(x = s \text{ and } y = t)$ or $(x = 1 - s \text{ and } \{y,t\} = \{0,1\}).$

2.2 Planar models and connected sums

The examples above are obtained by identifying edges of a square but we can use any polygon in the plane with an even number of sides to construct a closed surface so long as we prescribe the way to identify the sides in pairs. Drawing arrows then becomes tiresome so we describe the identification more systematically: going round clockwise we give each side a letter a say, and when we encounter the side to be identified we call it a if the arrow is in the same clockwise direction and a^{-1} if it is the opposite. For example, instead of



we call the top side a and the bottom b and get

 $aa^{-1}bb^{-1}$.

This is the sphere. Projective space is then abab, the Klein bottle $abab^{-1}$ and the torus $aba^{-1}b^{-1}$. Obviously the cyclic order is not important. There are lots of planar models which define the same surface. The sphere for example can be defined not just from the square but also by aa^{-1} , a 2-sided polygon:



and similarly the projective plane is aa.

Can we get new surfaces by taking more sides? Certainly, but first let's consider another construction of surfaces. If X and Y are two closed surfaces, remove a small closed disc from each. Then take a homeomorphism from the boundary of one disc to the boundary of the other. The topological space formed by identifying the two circles is also a surface called the *connected sum* X # Y. We can also think of it as joining the two by a cylinder:



The picture shows that we can get a surface with two holes from the connected sum of two tori. Let's look at this now from the planar point of view.

First remove a disc which passes through a vertex but otherwise misses the sides:



Now open it out:

and paste two copies together:



This gives an octagon, and the identification is given by the string of letters:

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}.$$

It's not hard to see that this is the general pattern: a connected sum can be represented by placing the second string of letters after the first. So in particular

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$$

describes a surface in \mathbb{R}^3 with g holes.

Note that when we defined a torus from a square, all four vertices are equivalent and this persists when we take the connected sum as above. The picture of the surface one should have then is 2g closed curves emanating from a single point, and the complement of those curves is homeomorphic to an open disc – the interior of the polygon.



If S is a sphere, then removing a disc just leaves another disc so connected sum with S takes out a disc and replaces it. Thus

$$X \# S = X.$$

Connected sum with the projective plane P is sometimes called *attaching a cross-cap*. In fact, removing a disc from P gives the Möbius band



so we are just pasting the boundary circle of the Möbius band to the boundary of the disc. It is easy to see then that the connected sum P # P is the Klein bottle.

You can't necessarily cancel the connected sum though: it is not true that X # A = Y # A implies X = Y. Here is an important example:

Proposition 2.1 The connected sum of a torus T and the projective plane P is homeomorphic to the connected sum of three projective planes.

Proof: From the remark above it is sufficient to prove that P#T = P#K where K is the Klein bottle. Now since P can be described by a 2-gon with relation *aa* and the Klein bottle is $bcbc^{-1}$, P#K is defined by a hexagon and the relation $aabcbc^{-1}$.



Now P # T is $aabcb^{-1}c^{-1}$:



Cut along the dotted line...



 \ldots detach the triangle and turn it over \ldots



... reattach...



... cut down the middle...



... turn the left hand quadrilateral over and paste together again...



...and this is $aabcbc^{-1}$.

2.3 The classification of surfaces

The planar models allow us to classify surfaces. We shall prove the following

Theorem 2.2 A closed, connected surface is either homeomorphic to the sphere, or to a connected sum of tori, or to a connected sum of projective planes.

We sketch the proof below (this is not examinable) and refer to [3] or [1] for more details. We have to start somewhere, and the topological definition of a surface is quite general, so we need to invoke a theorem beyond the scope of this course: any closed surface X has a triangulation: it is homeomorphic to a space formed from the disjoint union of finitely many triangles in \mathbb{R}^2 with edges glued together in pairs. For a Riemann surface (see next section), we can directly find a triangulation so long as we have a meromorphic function, and that is also a significant theorem, so we can't escape this starting point.

We shall proceed by using a planar model but there is also an alternative proof in [2] (or download from here: **new.math.uiuc.edu/zipproof/zipproof.pdf**) if you don't object to surfaces covered with zip fasteners.

Now take one triangle on the surface, and choose a homeomorphism to a planar triangle. Take an adjacent one and the common edge and choose a homeomorphism to another plane triangle and so on... Since the surface is connected the triangles form a polygon and thus X can be obtained from this polygon with edges glued together in pairs. It remains to systematically reduce this, without changing the homeomorphism type, to a standard form.

Step 1: Adjacent edges occurring in the form aa^{-1} or $a^{-1}a$ can be eliminated.



Step 2: We can assume that all vertices must be identified with each other. To see this, suppose Step 1 has been done, and we have two adjacent vertices in different

equivalence classes: red and yellow. Because of Step 1 the other side going through the yellow vertex is paired with a side elsewhere on the polygon. Cut off the triangle and glue it onto that side:



The result is the same number of sides but one less yellow and one more red vertex. Eventually, applying Step 1 again, we get to a single equivalence class.

Step 3: We can assume that any pair of the form a and a are adjacent, by cutting and pasting:



We now have a single equivalence class of vertices and all the pairs a, a are adjacent. What about a pair a, a^{-1} ? If they are adjacent, Step 1 gets rid of them, if not we have this:



If all the sides on the top part have their partners in the top part, then their vertices will never be equivalent to a vertex in the bottom part. But Step 2 gave us one equivalence class, so there is a b in the top half paired with something in the bottom.

It can't be b because Step 3 put them adjacent, so it must be b^{-1} .

Step 4: We can reduce this to something of the form $cdc^{-1}d^{-1}$ like this. First cut off the top and paste it to the bottom.



Now cut away from the left and paste it to the right.



Finally our surface is described by a string of terms of the form aa or $bcb^{-1}c^{-1}$: a connected sum of projective planes and tori. However, if there is at least one projective plane we can use Proposition 2.1 which says that P#T = P#P#P to get rid of the tori.

2.4 Orientability

Given a surface, we need to be able to decide what connected sum it is in the Classification Theorem without cutting it into pieces. Fortunately there are two concepts, which are invariant under homeomorphism, which do this. The first concerns orientation:

Definition 5 A surface X is orientable if it contains no open subset homeomorphic to a Möbius band.

From the definition it is clear that if X is orientable, any surface homeomorphic to X is too.

We saw that taking the connected sum with the projective plane means attaching a Möbius band, so the surfaces which are connected sums of P are non-orientable. We need to show that connected sums of tori are orientable. For this, we observe that the connected sum operation works for tori in \mathbf{R}^3 embedded in the standard way:



so a connected sum of tori can also be embedded in \mathbb{R}^3 . The sketch proof below assumes our surfaces are differentiable – we shall deal with these in more detail later.

Suppose for a contradiction that X is a non-orientable compact smooth surface in \mathbf{R}^3 . Then X has an open subset which is homeomorphic to a Möbius band, which means that we can find a loop (i.e. a closed path) in X such that the normal to X, when transported around the loop in a continuous fashion, comes back with the opposite direction. By considering a point on the normal a small distance from X, moving it around the loop and then connecting along the normal from one side of X to the other, we can construct a closed path $\gamma : [0,1] \to \mathbf{R}^3$ in \mathbf{R}^3 which meets X at exactly one point and is *transversal* to X at this point (i.e. the tangent to γ at x is not tangent to X). It is a general fact about the topology of \mathbf{R}^3 that any closed differentiable path $\gamma : [0,1] \to \mathbf{R}^3$ can be 'filled in' with a disc; more precisely there is a differentiable map $f: D \to \mathbf{R}^3$, where $D = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 \leq 1\}$, such that

$$\gamma(t) = f(\cos 2\pi t, \sin 2\pi t)$$

for all $t \in [0, 1]$. Now we can perturb f a little bit, without changing γ or the values of f on the boundary of D, to make f transversal to X (i.e. the image of f is not tangent to X at any point of intersection with X). But once f is transversal to Xit can be shown that the inverse image $f^{-1}(X)$ of X in D is very well behaved: it consists of a disjoint union of simple closed paths in the interior of D, together with paths meeting the boundary of D in exactly their endpoints (which are two distinct points on the boundary of D). Thus $f^{-1}(X)$ contains an even number of points on the boundary of D. The surface must therefore be orientable. This argument shows why the projective plane in particular can't be embedded in \mathbf{R}^3 . Here is an amusing corollary:

Proposition 2.3 Any simple closed curve in the plane contains an inscribed rectangle.



Proof: The closed curve C is homeomorphic to the circle. Consider the set of pairs of points (x, y) in C. This is the product of two circles: a *torus*. We now want to consider the set X of *unordered* pairs, so consider the planar model of the torus. We identify (x, y) with (y, x), which is reflection about the diagonal. The top side then gets identified with the right hand side, and under the torus identification with the left hand side.



The set of unordered points is therefore obtained by identification on the top triangle:



and this is the projective plane with a disc removed (the Möbius band):



Now define a map $f: X \to \mathbf{R}^3$ as follows:

$$(x,y) \mapsto (\frac{1}{2}(x+y), |x-y|) \in \mathbf{R}^2 \times \mathbf{R}$$

The first term is the midpoint of the line xy and the last is the distance between x and y. Both are clearly independent of the order and so the map is well-defined. When x = y, which is the boundary circle of the Möbius band, the map is

$$x \mapsto (x,0)$$

which is the curve C in the plane $x_3 = 0$. Since the curve bounds a disc we can extend f to the surface obtained by pasting the disc to X and extending f to be the inclusion of the disc into the plane $x_3 = 0$. This is a continuous map (it can be perturbed to be differentiable if necessary) of the projective plane P to \mathbb{R}^3 . Since Pis unorientable it can't be an embedding so we have at least two pairs $(x_1, y_1), (x_2, y_2)$ with the same centre and the same separation. These are the vertices of the required rectangle.

2.5 The Euler characteristic

It is a familar fact (already known to Descartes in 1639) that if you divide up the surface of a sphere into polygons and count the number of vertices, edges and faces then



This number is the Euler characteristic, and we shall define it for any surface. First we have to define our terms:

Definition 6 A subdivision of a compact surface X is a partition of X into

i) vertices (these are finitely many points of X),

ii) edges (finitely many disjoint subsets of X each homeomorphic to the open interval (0,1)), and

iii) faces (finitely many disjoint open subsets of X each homeomorphic to the open disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in \mathbb{R}^2 ,

such that

a) the faces are the connected components of $X \setminus \{vertices and edges\},\$

b) no edge contains a vertex, and

c) each edge 'begins and ends in a vertex' (either the same vertex or different vertices), or more precisely, if e is an edge then there are vertices v_0 and v_1 (not necessarily distinct) and a continuous map

$$f: [0,1] \to e \cup \{v_0, v_1\}$$

which restricts to a homeomorphism from (0,1) to e and satisfies $f(0) = v_0$ and $f(1) = v_1$.

Definition 7 The Euler characteristic (or Euler number) of a compact surface X with a subdivision is

$$\chi(X) = V - E + F$$

where V is the number of vertices, E is the number of edges and F is the number of faces in the subdivision.

The fact that a closed surface has a subdivision follows from the existence of a triangulation. The most important fact is

Theorem 2.4 The Euler characteristic of a compact surface is independent of the subdivision

which we shall sketch a proof of later.

A planar model provides a subdivision of a surface. We have one face – the interior of the polygon – and if there are 2n sides to the polygon, these get identified in pairs so there are n edges. For the vertices we have to count the number of equivalence classes, but in the normal form of the classification theorem, we created a single equivalence class. In that case, the Euler characteristic is

$$1 - n + 1 = 2 - n.$$

The connected sum of g tori had 4g sides in the standard model $a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}$ so in that case $\chi(X) = 2 - 2g$. The connected sum of g projective planes has 2g sides so we have $\chi(X) = 2 - g$. We then obtain:

Theorem 2.5 A closed surface is determined up to homeomorphism by its orientability and its Euler characteristic.

This is a very strong result: nothing like this happens in higher dimensions.

To calculate the Euler characteristic of a given surface we don't necessarily have to go to the classification. Suppose a surface is made up of the union of two spaces Xand Y, such that the intersection $X \cap Y$ has a subdivision which is a subset of the subdivisions for X and for Y. Then since V, E and F are just counting the number of elements in a set, we have immediately that

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y).$$

We can deal with a connected sum this way. Take a closed surface X and remove a disc D to get a space X^o . The disc has Euler characteristic 1 (a polygon has one face,

n vertices and n sides) and the boundary circle has Euler characteristic 0 (no face). So applying the formula,

$$\chi(X) = \chi(X^o \cup D) = \chi(X^o) + \chi(D) - \chi(X^o \cap D) = \chi(X^o) + 1.$$

To get the connected sum we paste X^o to Y^o along the boundary circle so

$$\chi(X \# Y) = \chi(X^o) + \chi(Y^o) - \chi(X^o \cap Y^o) = \chi(X) - 1 + \chi(Y) - 1 - 0 = \chi(X) + \chi(Y) - 2.$$

In particular, $\chi(X\#T) = \chi(X) - 2$ so this again gives the value 2 - 2g for the connected sum of g tori.

To make all this work we finally need:

Theorem 2.6 The Euler characteristic $\chi(X)$ of a compact surface X is a topological invariant.

We give a sketch proof (which is not examinable).

Proof:

The idea is to give a different definition of $\chi(X)$ which makes it clear that it is a topological invariant, and then prove that the Euler characteristic of any subdivision of X is equal to $\chi(X)$ defined in this new way.

For each continuous path $f : [0,1] \to X$ define its boundary ∂f to be the formal linear combination of points f(0) + f(1). If g is another map and g(0) = f(1) then, with coefficients in $\mathbb{Z}/2$, we have

$$\partial f + \partial g = f(0) + 2f(1) + g(1) = f(0) + g(1)$$

which is the boundary of the path obtained by sticking these two together. Let C_0 be the vector space of finite linear combinations of points with coefficients in $\mathbb{Z}/2$ and C_1 the linear combinations of paths, then $\partial : C_1 \to C_0$ is a linear map. If X is connected then any two points can be joined by a path, so that $x \in C_0$ is in the image of ∂ if and only if it has an even number of terms.

Now look at continuous maps of a triangle $ABC = \Delta$ to X and the space C_2 of all linear combinations of these. The boundary of $F : \Delta \to X$ is the sum of the three paths which are the restrictions of F to the sides of the triangle. Then

$$\partial \partial F = (F(A) + F(B)) + (F(B) + F(C)) + (F(C) + F(A)) = 0$$

so that the image of $\partial : C_2 \to C_1$ is contained in the kernel of $\partial : C_1 \to C_0$. We define $H_1(X)$ to be the quotient space. This is clearly a topological invariant because we only used the notion of continuous functions to define it.

If we take X to be a surface with a subdivision, one can show that because each face is homeomorphic to a disc, any element in the kernel of $\partial : C_1 \to C_0$ can be replaced by adding on something in ∂C_2 by a linear combination of edges of the subdivision:



Now we let \mathcal{V} , \mathcal{E} and \mathcal{F} be vector spaces over $\mathbb{Z}/2$ with bases given by the sets of vertices, edges and faces of the subdivision, then define boundary maps in the same way

$$\partial : \mathcal{E} \to \mathcal{V} \text{ and } \partial : \mathcal{F} \to \mathcal{E}.$$

Then

$$H_1(X) \cong \frac{\ker(\partial : \mathcal{E} \to \mathcal{V})}{\operatorname{im}(\partial : \mathcal{F} \to \mathcal{E})}$$

By the rank-nullity formula we get

$$\dim H_1(X) = \dim \mathcal{E} - \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V}) - \dim \mathcal{F} + \dim \ker(\partial : \mathcal{F} \to \mathcal{E})$$

Because X is connected the image of $\partial : \mathcal{E} \to \mathcal{V}$ consists of sums of an even number of vertices so that

$$\dim \mathcal{V} = 1 + \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V})$$

Also $\ker(\partial: \mathcal{F} \to \mathcal{E})$ is clearly spanned by the sum of the faces, hence

$$\dim \ker(\partial : \mathcal{F} \to \mathcal{E}) = 1$$

 \mathbf{SO}

$$\dim H_1(X) = 2 - V + E - F.$$

This shows that V - E + F is a topological invariant.