Lecture 13:
The Cauchy-Riemann Equations

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13.1 Necessity of the Cauchy-Riemann Equations

Suppose f is differentiable at a point $z_0 \in \mathbb{C}$. If we write $z = x + iy$, $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$, and

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

then

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= (u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)) \\ &\quad - (u(x_0, y_0) + iv(x_0, y_0)) \\ &= (u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_)) \\ &\quad + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_)). \end{aligned}$$

Hence

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} \\ &\quad + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta z}. \end{aligned}$$

If we let $\Delta z \to 0$ along the real axis, then $\Delta z = \Delta x$, and we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= u_x(x_0, y_0) + i v_x(x_0, y_0).
\end{aligned}
$$

If we let $\Delta z \to 0$ along the imaginary axis, then $\Delta z = i\Delta y$, and we have

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_{)} + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
&= -i u_y(x_0, y_0) + v_y(x_0, y_0).
\end{aligned}
$$

It follows that

$$
u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).
$$

We call these equations the *Cauchy-Riemann equations*.

Theorem 13.1. Suppose f is differentiable at $z_0 = x_0 + iy_0$ and

$$
f(x + iy) = u(x, y) + iv(x, y).
$$

Then

$$
u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0)
$$

and

$$
f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).
$$

Example 13.1. If $f(z) = |z|^2$, then, writing $f(x + iy) = u(x, y) + iv(x, y)$,

$$
u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0.
$$

Hence

$$
u_x(x, y) = 2x, u_y(x, y) = 2y, \text{ and } v_x(x, y) = 0 = v_y(x, y).
$$

Hence f satisfies the Cauchy-Riemann equations only at the origin, showing that, as we have seen before, f is not differentiable at any $z \neq 0$.

13.2 Sufficiency of the Cauchy-Riemann equations

By themselves, the Cauchy-Riemann equations are not sufficient to guarantee the differentiability of a given function. However, the additional assumption of continuity of the partial derivatives does suffice to guarantee differentiability.

Theorem 13.2. Suppose f is defined on an ϵ neighborhood U of a point $z_0 = x_0 + y_0$,

$$f(x+iy) = u(x,y) + iv(x,y),$$

and u_x, u_y, v_x, and v_y exist on U and are continuous at (x_0, y_0). If

$$u_x(x_0,y_0) = v_y(x_0,y_0) \text{ and } u_y(x_0,y_0) = -v_x(x_0,y_0),$$

then f is differentiable at z_0.

Proof. Let $\Delta z = \Delta x + i\Delta y$,

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0),$$

and

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

The key fact we need comes from a theorem of multi-variable calculus: under the conditions of the theorem, u and v are both differentiable functions. This means that there exist ϵ_1 and ϵ_2, depending on Δx and Δy, such that

$$\Delta u = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \epsilon_1|\Delta z|$$

and

$$\Delta v = v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \epsilon_2|\Delta z|$$

and both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta z \to 0$. It follows that

$$\begin{aligned}
f(z_0+\Delta z) - f(z_0) &= \Delta u + i\Delta v \\
&= u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \epsilon_1|\Delta z| \\
&\qquad + i(v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \epsilon_2|\Delta z|) \\
&= u_x(x_0,y_0)\Delta x - v_x(x_0,y_)\Delta y + iv_x(x_0,y_0)\Delta x \\
&\qquad + iu_x(x_0,y_0)\Delta y + (\epsilon_1 + i\epsilon_2)|\Delta z| \\
&= u_x(x_0,y_0)\Delta z + iv_x(x_0,y_0)\Delta z + (\epsilon_1 + i\epsilon_2)|\Delta z|.
\end{aligned}$$

Hence

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0) + i v_x(x_0, y_0) + (\epsilon_1 + i\epsilon_2) \frac{|\Delta z|}{z} \right) \\ &= u_x(x_0, y_0) + i v_x(u_0, y_0), \end{aligned}$$

since

$$\lim_{\Delta z \to 0} (\epsilon_1 + \epsilon_2) = 0$$

and

$$\left| \frac{|z|}{z} \right| = \frac{|z|}{|z|} = 1$$

for all z. $\square$

Example 13.2. It follows now from the previous example that $f(z) = |z|^2$ is differentiable at $z = 0$ but not for any $z \neq 0$.

Example 13.3. If $f(z) = e^z$, then

$$f(x + iy) = e^x e^{iy} = e^x \cos(x) + i e^x \sin(y).$$

Hence

$$u(x, y) = e^x \cos(y) \text{ and } v(x, y) = e^x \sin(y).$$

Thus

$$u_x(x, y) = e^x \cos(y) = v_y \text{ and } u_y(x, y) = -e^x \sin(y) = -v_x(x, y).$$

Hence f is differentiable at every $z \in \mathbb{C}$. Moreover,

$$f'(z) = u_x(x, y) + i v_x(x, y) = e^x \cos(y) + i e^x \sin(y) = e^z = f(z).$$